Augmenting Batch Exchanges with Constant Function Market Makers

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Outline

- Two ideas in exchange design with newfound popularity
- How should we combine them?
- Goal: Map out design space (no dominant design)

Exchange Model

- **Users trade N divisible, fungible assets through *limit orders***
  - “Sell 1 unit of $\mathcal{X}$ for at least 2 units of $\mathcal{Y}$”

Two Exchange Design Innovations

Batch Exchanges

- **Execute batches of trades, all at once**
- **Input: Set of limit orders**

1. **Compute Prices**
2. **Trade in batch at price quotients**
    - Meaningless units
    - No pairwise matching

- **“Clearing” if no debt**

Sell 10 USD for EUR
$\mathbf{min}\ \frac{9}{10}\frac{\text{EUR}}{\text{USD}}$

Sell 9 EUR for JPY
$\mathbf{min}\ 140\frac{\text{JPY}}{\text{EUR}}$

Sell 1350 JPY for USD
$\mathbf{min}\ \frac{1}{135}\frac{\text{USD}}{\text{JPY}}$

Sell 10000 USD for EUR
$\mathbf{min}\ 1000\frac{\text{EUR}}{\text{USD}}$

Pricing Engine

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Pricing Engine

$$p_{\text{USD}} = 9$$
$$p_{\text{EUR}} = 10$$
$$p_{\text{JPY}} = \frac{1}{15}$$

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$$p_{\text{USD}} = 9$$
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€9 $\Leftrightarrow$ \$10

\$10 $\Leftrightarrow$ ¥1350

¥1350 $\Leftrightarrow$ €9

Two Exchange Design Innovations

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Sell 10 USD for EUR
$\min \frac{9}{10}\frac{\text{EUR}}{\text{USD}} \leq \frac{p_{\text{USD}}}{p_{\text{EUR}}} = \frac{9}{10}$ ✓

Sell 9 EUR for JPY
$\min 140\frac{\text{JPY}}{\text{EUR}} \leq \frac{p_{\text{EUR}}}{p_{\text{JPY}}} = 150$ ✓

Sell 1350 JPY for USD
$\min \frac{1}{135}\frac{\text{USD}}{\text{JPY}} \leq \frac{p_{\text{JPY}}}{p_{\text{USD}}} = \frac{1}{135}$ ✓

Sell 10000 USD for EUR
$\min 1000\frac{\text{EUR}}{\text{USD}} \not\leq \frac{p_{\text{USD}}}{p_{\text{EUR}}} = \frac{9}{10}$ ✗

Pricing Engine

$$p_{\text{USD}} = 9$$
$$p_{\text{EUR}} = 10$$
$$p_{\text{JPY}} = \frac{1}{15}$$

€9 ⇔ \$10

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¥1350 ⇔ €9

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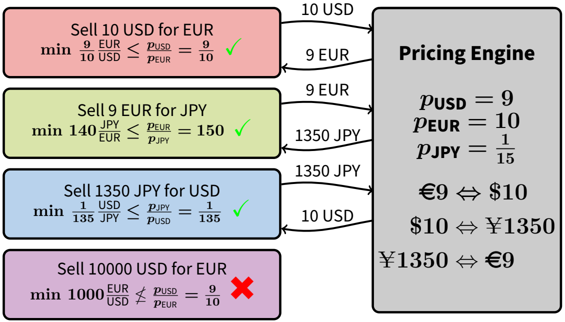
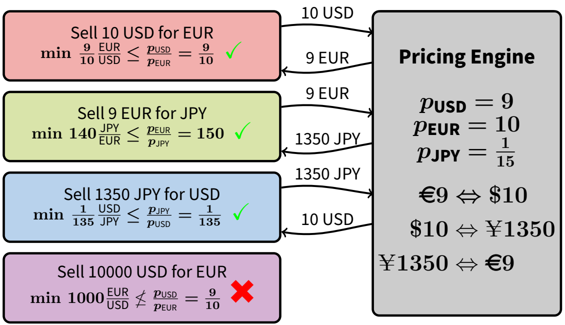


Theorem (Arrow and Debreu, 1954)

$\exists$ *unique* equilibrium prices* $\{p_A\}$ *and allocations that clear the market.*

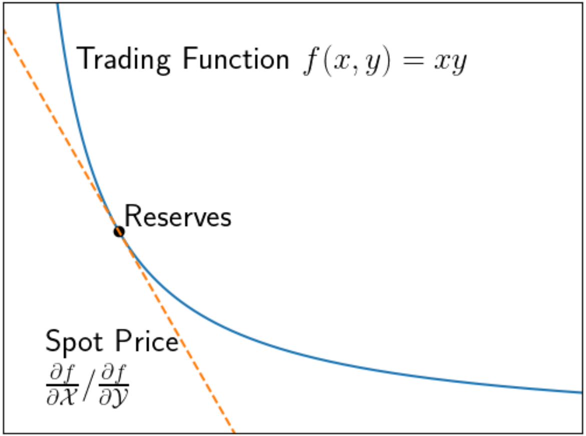
Key Properties of Batch Exchange Model

1. **Uniform prices (unique!) bring economic benefits**
   - Pareto-Optimal (for limit orders)
   - E.g. Budish et al. "The high-frequency trading arms race" (2015)
2. **Requires computing Arrow-Debreu exchange market equilibria**



Two Exchange Design Innovations
Constant Function Market Makers

- **CFMM maintains reserves and a *trading function* $f(\cdot)$**
- **Accepts trade from (x, y) to (x', y') if and only if $f(x, y) \leq f(x', y')$**
- **Why?**
  - Market-makers add liquidity
  - Automated
  - Computational simplicity



Our Work In Context

- **Several projects combine batch exchanges with CFMMs, using different mechanisms**
  - Penumbra, CoWSwap, [Walther, 2021], [Canidio and Fritsch, 2023]
- **What are the tradeoffs for different mechanisms for integrating CFMMs into batch exchanges?**

Augmenting Batch Exchanges with CFMMs

How can batch exchanges draw on passive liquidity?

- **Model:**
  - N assets $\mathcal{X} \in \mathfrak{A}$
  - 1 batch exchange
  - Many CFMMs, with different curves, reserves
  - Also outside world—other exchanges, other users, ...

Augmenting Batch Exchanges with CFMMs

How can batch exchanges draw on passive liquidity?

- **Axiom 1: Asset conservation**
- **Axiom 2: Uniform Prices** $\{p_{\mathcal{X}}\}_{\mathcal{X}\in\mathfrak{A}}$
  - No trade from $\mathcal{X}$ to $\mathcal{Y}$ gets a better rate than $\frac{p_{\mathcal{X}}}{p_{\mathcal{Y}}}$.

Augmenting Batch Exchanges with CFMMs

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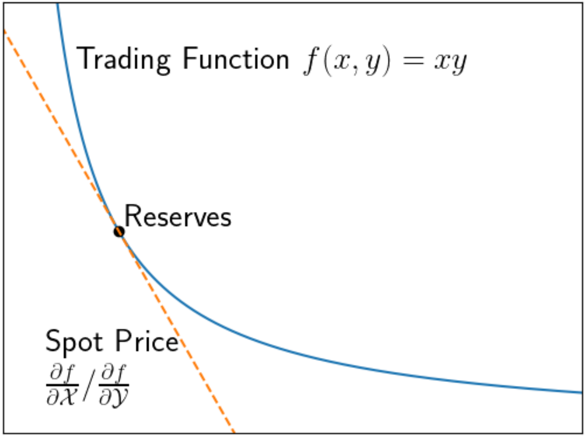
  - No trade from $\mathcal{X}$ to $\mathcal{Y}$ gets a better rate than $\frac{p_{\mathcal{X}}}{p_{\mathcal{Y}}}$.
- **Axiom 3: Limit orders make best responses**
  - A limit order trades $\mathcal{X}$ to $\mathcal{Y}$ at no worse than the market rate $\frac{p_{\mathcal{X}}}{p_{\mathcal{Y}}}$, only if market rate exceeds limit price
- **These are standard market design assumptions, lead to classic theory results on Arrow-Debreu market equilibria.**

How can batch exchanges draw on passive liquidity?

- **Axiom 4: CFMM trading function must not decrease**



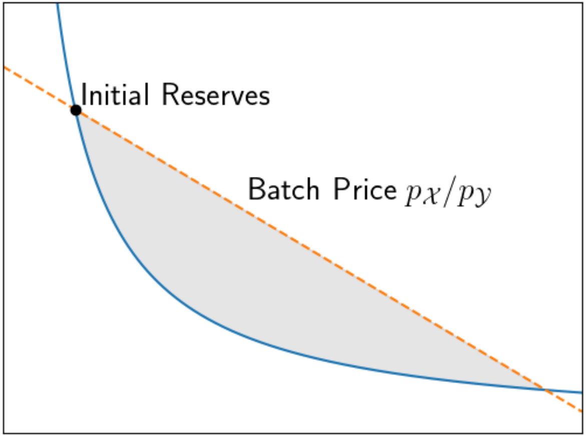
CFMMs in Batch Exchanges

How can batch exchanges draw on passive liquidity?

- **Axiom 4: CFMM trading function must not decrease**

Consequence

Market equilibrium is no longer unique



CFMMs in Batch Exchanges

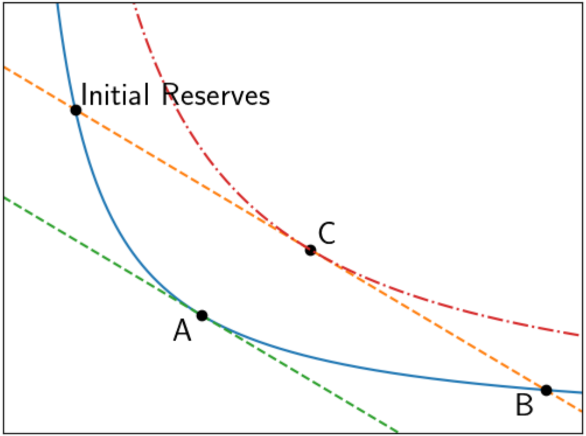
How can batch exchanges draw on passive liquidity?

- **Axiom 4: CFMM trading function must not decrease**

Consequence

Market equilibrium is no longer unique

- **How should a batch choose a CFMM's trade?**
- **Also complicates equilibria computation**

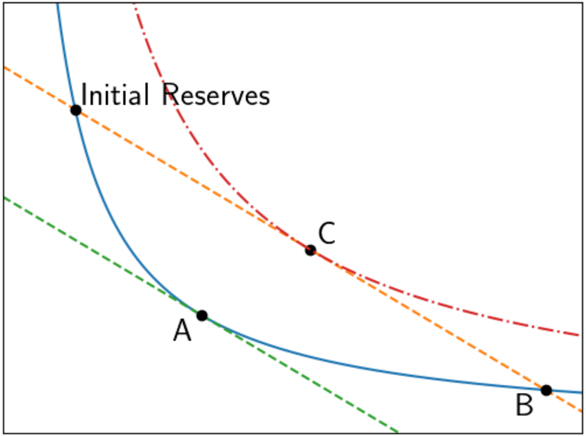


CFMMs in Batch Exchanges

- **Asset Conservation and Uniform Prices imply:**

Consequence

CFMMs must trade *at* market prices, not below



Some Desirable Properties

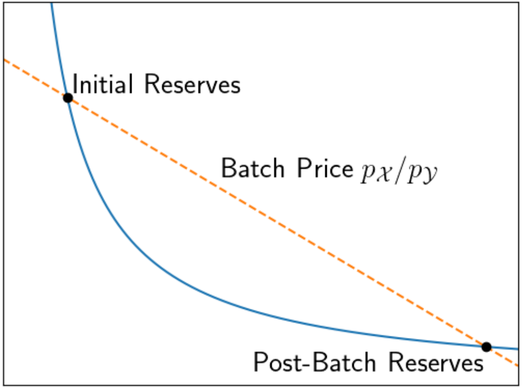
- **Pareto Optimality**
  - From perspective of limit orders
  - Recall: Without CFMMs, every equilibrium is Pareto Optimal
- **Price Coherence**
  - After a batch, CFMM spot exchange rates are quotients of some set of prices
  - Otherwise, cyclic arbitrage opportunity (free money)
- **Preservation of Price Coherence**
  - Price coherence, but only if prices are also coherent before batch

Consequences

Consequence

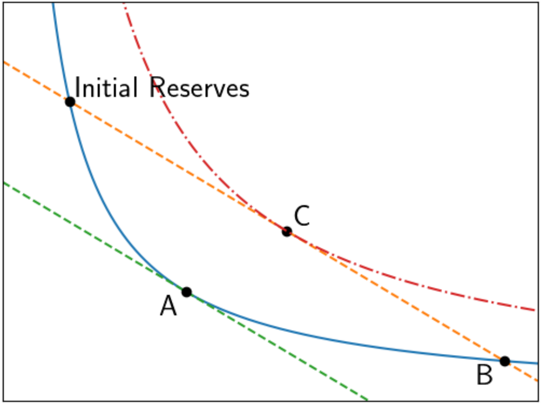
No mechanism can in all circumstances guarantee Pareto Optimality and (Preservation of) Price Coherence

- **Proof Intuition:**
  - PO can require trading all the way across
  - Multiple CFMMs with different curves will end at different spot exchange rates



Some More Desiderata and Consequences

- **Joint Price Discovery (JPD)**
  - After a batch, CFMM spot prices equal batch prices
  - Prevents a common atomic, risk-free "cyclic" arbitrage
- **JPD requires maximizing $f(\cdot)$ (trading to C)**
- **An example of how context matters:**
  - Trading to C incentivizes splitting trade over many batches, but trading to B does not.
  - How are batches initiated?
  - How many users?

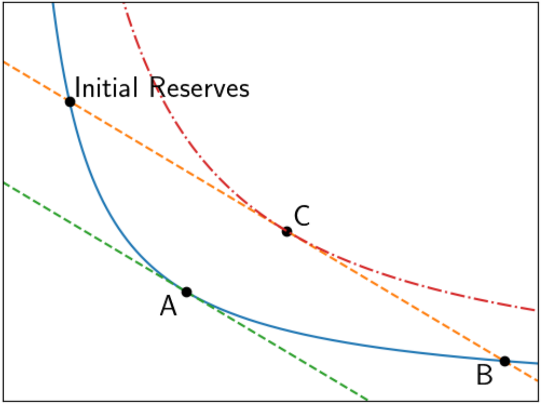


Some More Desiderata and Consequences

- **Locally Computable Rule (LCR)**
  - CFMM trade depends only on CFMM state and market price

Consequence

Trading to *C* is a LCR that satisfies Price Coherence

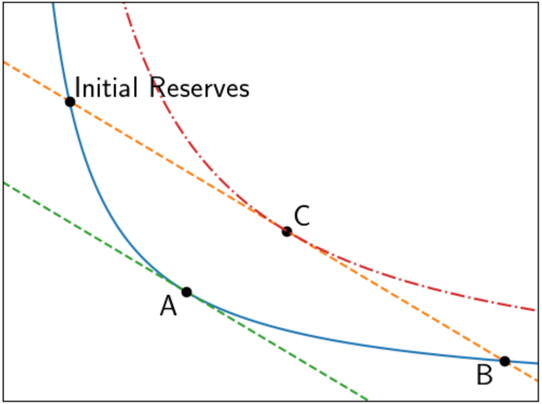


Some More Desiderata and Consequences

Consequence

Trading to B is a LCR that guarantees Preservation of Price Coherence, if and only if all CFMMs use a *constant product* curve.

- ***Unique* exception to incompatibility between PO and PPC**



Computing Equilibria

- **Mixed-Integer Programs [Walther21] or general (not always convex) solvers**
- **LCR $\Rightarrow$ algorithms based on auctions, iterations (Tâtonnement) are directly applicable**
  - LCR must satisfy Weak Gross Substitutability
  - Price goes up $\implies$ demand does not increase
- **What about other approaches? Convex programs?**

A Convex Program for 2-Asset WGS Utility Functions

Observation

A CFMM trading between 2 assets, with a LCR satisfying WGS, acts like an (uncountably) infinite set of infinitesimal limit orders.

Let's adapt a convex program for linear Arrow-Debreu exchange markets [DGV16] to support CFMMs trading between 2 assets

A Convex Program for Linear Utility Functions [DGV16]

$$\text{Minimize} \sum_i p_i \left(e_i \ \ln(\frac{p_i}{\beta_i}) \right) - \sum_i y_{i,j} \ln u_{i,j}$$

$$\text{Subject to} \sum_i y_{i,j} = \sum_i y_{j,i} \qquad \forall j \in [N]$$

$$p_j \geq 1 \qquad \forall j \in [N]$$

$$y_i \geq 0 \qquad \forall i \in [M]$$

$$u_{i,j}\beta_i \leq p_j \qquad \forall i,j$$

A Convex Program for 2-asset WGS CFMM Trading Functions

$$\text{Minimize} \sum_i p_{A_i} \int_0^\infty \left(d_i(z) \ln\left(\frac{p_{A_i}}{\beta_{i,z}(p)}\right)\right) dz - \sum_i p_{A_i} g_i(y_i/p_{A_i})$$

$$\text{Subject to} \sum_{i:A_i=j} y_i = \sum_{i:B_i=j} y_i \qquad \forall j \in [N]$$

$$p_j \geq 1 \qquad \forall j \in [N]$$

$$y_i \geq 0 \qquad \forall i \in [M].$$

Equivalently, this program solves exchange markets where each agent is interested in only two assets, using *any* WGS utility function on those two assets.

Conclusion

- **Axiomatic framework for integrating CFMMs into batch exchanges**
  - Extra degree of freedom requires deliberate choice
- **Natural desiderata are incompatible**
  - Pareto-Optimality at odds with Price Coherence
- **Convex program for exchange markets with 2-asset WGS CFMMs**